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Hardy Function and Unique Continuation for Evolution Equations

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In this paper, unique continuation problems are considered for the first order evolution equation:

$$u_t(x,t) = P(t, D)\, u(x,t), \qquad x \in R^n, \quad t \in R, \tag{$*$}$$

where $P(t, D)$ represents a mth order differential operator with time dependent coefficients. In one space dimension case ($n = 1$), a necessary and sufficient condition is given for any nonzero solution of $(*)$ in the class of $C(R; L^2(R))$ to have a support on a horizontal half line in the x–t space at two different times. With some assumptions on the coefficients of $P(t, D)$, it is shown that any solution $u(x,t) \in C(R; L^2(R))$ is uniquely determined by its part on any two horizontal half lines or any open subset in the x–t space. In higher space dimension case ($n > 1$), a necessary condition is given for any solution $u \in C(R; L^2(R^n))$ of $(*)$ to be supported on a half hyperplane in the space R^n at two different times. Some higher order evolution equations are also considered for their unique continuation properties.

1. INTRODUCTION

In this paper, we consider unique continuation problems of the evolution equation:

$$u_t(x,t) = P(t, D)\, u(x,t), \qquad x \in R, \quad t \in R \tag{1.1}$$

where the symbol $P(t, D)$ represents a mth order differential operator

$$P(t, D) = \sum_{j=0}^{m} a_j(t)\, D^j, \qquad m \geqslant 1$$

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with $D = (1/i)\,\partial_x$, $i = \sqrt{-1}$, and subscripts t and x denote partial differentiation. The coefficients $a_j(t)$ of $P(t, D)$ are assumed to be complex value functions of t and $a_j(t) \in L^1_{\mathrm{loc}}(R)$.

Let $u(x, t) \in C(R; L^2(R))$ be a solution of (1.1) (in the weak sense) and suppose that $u(x, t)$ is supported on a horizontal half line in the x-t space at time $t = t_0$ for some $t_0 \in R$. We shall show it is necessary that

$$\int_{t_0}^{t_1} a_j(t)\, dt = 0 \qquad \text{for all} \quad j \geqslant 2,$$

in order that the solution $u(x, t)$ is supported on a horizontal half line in the x-t space at another time $t = t_1$. Based on this result, with a few more assumptions on the coefficients $a_j(t)$ of $P(t, D)$, we will show that any solution $u(x, t)$ of (1.1) in the class of $C(R; L^2(R))$ is uniquely determined by its part on any two different horizontal half lines in the x-t space or its part on any open subset in the x-t space. That is to say, if we have two solutions $u(x, t)$ and $v(x, t)$ of (1.1) and they agree with each other on two horizontal half lines or some open subset in the x-t space, then $u(x, t)$ and $v(x, t)$ must identically equal to each other in the x-t space. The same results will also be established for the second order evolution equation

$$u_{tt}(x, t) = \alpha D^{(2m)} u(x, t), \qquad x \in R, \quad t \in R$$

where $m \geqslant 2$ and α is a complex constant.

In addition, in this paper, we shall consider evolution equation in higher space dimension as follows:

$$u_t(x, t) = \sum_{|p| \leqslant m} a_p(t)\, D^p u(x, t), \qquad x \in R^n, \quad t \in R, \tag{1.2}$$

where $p = (p_1, p_2, ..., p_n)$ is a multi-index, the p_j are nonnegative integers, $|p| = \sum_{j=1}^{m} p_j$ and

$$D^p = D_1^{p_1} D_2^{p_2} \cdots D_n^{p_n}$$

with $D_j = (1/i)\,\partial_{x_j}$, $j = 1, 2, ..., n$. The coefficients $a_p(t)$ are complex value functions of t and $a_p(t) \in L^1_{\mathrm{loc}}(R)$.

A necessary condition will be given for the solutions of (1.2) in the class of $C(R; L^2(R^n))$ such that they are supported on a half hyperplane in the space R^n at two different moments. It follows from this result, as a directly application, that the only possible evolution equation of form (1.2), which has finite speed of propagation, is the equation that $P(t, D)$ is a first order differential operator. If $P(t, D)$ is a differential operator of order $m \geqslant 2$, then evolution equation (1.2) has infinite speed of propagation. Namely, if $u(x, t) \in C(R; L^2(R))$ is a solution of (1.2) and has compact support in the space R^n at time $t=t_0$, then there is a $t_1>t_0$ such that the support of $u(x,t)$ in the space R^n at time $t=t_1$ is not compact (see [22]).

We shall see similar results for a class of higher order evolution equations.

The study of unique continuation property for partial differential operators began with Carleman's work ([2]) in which he proved that the operator $L=\Delta+v(x)$ in R^2 with $v\in L^\infty_{\mathrm{loc}}(R^2)$ has the strong unique continuation property, i.e., $u\equiv 0$ is the only solution to $Lu=0$ which has infinite zero at a point in R^2. Since then, many unique continuation results have been obtained for elliptic operators (cf. [7], [8] and [16] as well as many references contained in these papers). As regards evolution equations, Mizohata ([12]), Saut and Scheurer ([17]) showed that a class of second order parabolic operator L with variable coefficients, defined on a connected open set $\Omega\subset R_t\times R^n_x$, has the unique continuation property that every solution u of $Lu=0$ which vanishes on an open subset Q of Ω vanishes in the horizontal component of Q in Ω i.e., in $\{(x,t)\in\Omega;$ there exits x_1, $(x_1,t)\in Q\}$. Some higher order parabolic operators and some dispersive operators in one space dimension are also considered by Saut and Scheurer in [17]. Kenig and Sogge [11] considered Schrödinger operator and they proved the following global unique continuation property:

If $u(x,t)\in H^{2(n+1)/(n+4)}(R^{n+1})$ satisfies the differential inequality

$$\left|\left(i\frac{\partial}{\partial t}+\Delta\right)u(x,t)\right|\leqslant|v(x,t)\,u(x,t)|$$

for some $v(x,t)\in L^{(n+2)/2}(R^{n+1})$, then $u(x,t)$ must vanish identically if it vanishes in a half space in R^{n+1}.

Similar results are obtained for the hyperbolic operator $\delta_{tt}-\Delta$ in [10] by Kenig, Ruiz and Sogge. We refer to [6], [14] and [19] for more results on this subject.

The major method to study unique continuation properties of partial differential operators is to establish the corresponding Carleman type inequality, which is originated from Carleman's pioneering work [2]. However, our approach is quite different and is mainly based on some basic facts of Hardy functions. The key ingredient of our approach is an estimate of Hardy function in the class of H^2_+. More precisely, let $z(t)$ be a continuous curve in the upper half complex plane with $z(t)\to\infty$ as $t\to\infty$. Then for any $0\neq h(z)\in H^2_+$ there exist a $\delta>0$ and a sequence of $\{t_n\}$ with $t_n\to\infty$ such that

$$|h(z(t_n))|\geqslant\exp\left\{-\delta\frac{|z(t_n)|^2}{\operatorname{Im} z(t_n)}\right\}$$

for $n=1,2,\ldots$ This estimate has been proved in our earlier paper [22]. Based on this estimate, it is quite elementary to obtain the unique continuation properties we have described for evolution equation (1.1) and (1.2).

This paper is organized as follows.

—In Section 2, we recall some basic definitions and facts of Hardy function. An estimate of decaying rate of Hardy function along any ray in the upper half complex plane is given.

—In Section 3, we consider evolution equation (1.1) in one space dimension. Based on the estimate established in section 2 and the Fourier transform, various unique continuation results are proved for evolution equation (1.1).

—In Section 4, evolution equation (1.2) is discussed. In addition, some remarks are given for the further investigation.

2. Estimates of Hardy Functions

We use the following notations for the Fourier transform and the inverse Fourier transform:

$$\hat{f}(\lambda)=\int_{-\infty}^{+\infty} e^{ix\lambda}f(x)\,dx$$

and

$$\check{f}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ix\lambda}f(\lambda)\,d\lambda.$$

By H^2_+ we denote the Hardy space of functions $h(k)$ analytic in $\operatorname{Im} k>0$ with

$$\sup_{b>0}\int_{-\infty}^{+\infty} |h(a+ib)|^2\,da<+\infty. \tag{2.1}$$

Such a function $h(k)$ assumes boundary values

$$\lim_{\varepsilon\to 0^+} h(a+i\varepsilon)$$

in $L^2(R)$.

Similarly, we denote by H^2_- the Hardy space of functions analytic in $\operatorname{Im} k<0$ with

$$\sup_{b<0}\int_{-\infty}^{+\infty} |h(a+ib)|^2\,da<+\infty.$$

It is well-known [5] that $h \in H^2_+$ if and only if there exists an $f \in L^2(R)$ with support of $f \subset (0, +\infty)$ such that

$$h(k) = \int_{-\infty}^{+\infty} e^{ikx} f(x)\, dx = \int_0^{\infty} e^{ikx} f(x)\, dx. \tag{2.2}$$

From (2.1) and (2.2), we have

LEMMA 2.1. *If $h(k) \in L^2(R)$ and the support of $\check{h} \subset (0, \infty)$, then $h(k)$ can be extended to the upper half complex plane as an analytic function and*

$$\sup_{b>0} \int_{-\infty}^{+\infty} |h(a+ib)|^2\, da < +\infty.$$

The following lemma, which gives a lower bound estimate for any $h(k) \in H^2_+$, is a key ingredient in the proof of our main results.

LEMMA 2.2. *If $0 \neq h \in H^2_+$, then for any continuous curve*

$$s(\tau) = u(\tau) + iv(\tau), \qquad 0 \leqslant \tau < +\infty$$

in the upper half complex plane with

$$\lim_{\tau \to +\infty} s(\tau) = \infty,$$

there exists a $\delta > 0$ and a sequence of $\{\tau_n\}$ with

$$\lim_{n \to +\infty} \tau_n = +\infty$$

such that

$$|h(s(\tau_n))| \geqslant \exp\left\{ -\delta \frac{u^2(\tau_n) + v^2(\tau_n)}{v(\tau_n)} \right\}$$

for $n = 1, 2, 3, \ldots$

The complete proof of this lemma has been given in [22]. However, because of its importance in this paper and also for convenience of the reader, we have attached the proof as an appendix at the end of the paper.

The following lemma is a special case of Lemma 2.2.

LEMMA 2.3. *If $0 \neq h(z) \in H^2_+$, then for any $a \in R$ there exists a $\delta > 0$ and a sequence of $\{b_n\}$ with $b_n \to +\infty$ such that*

$$|h((a+i)\, b_n)| \geqslant e^{-\delta b_n}, \qquad n = 1, 2, \ldots$$

Finally, we prove a technical lemma to end this section.

LEMMA 2.4. *For any integer $m \geqslant 2$, there exists a $0 \neq \beta_m \in R$ such that*

$$\operatorname{Re}\{\alpha(\beta_m + i)^m\} \equiv c_m > 0. \qquad (\alpha \neq 0)$$

Proof. If $m = 2$, we write α as $\alpha = a + ib$. Then

$$c_2 \equiv \operatorname{Re}\{\alpha(\beta_2 + i)^2\} = a\beta_2^2 - 2b\beta_2 - a.$$

It is obvious that there is a $\beta_2 \in R$ such that $c_2 > 0$.

For the case where $m \geqslant 3$, we write α as

$$\alpha = |\alpha|\, e^{i\theta}, \qquad 0 \leqslant \theta < 2\pi,$$

and $\beta_m + i$ as

$$\beta_m + i = \sqrt{\beta_m^2 + 1}\,(\cos\mu + i\sin\mu).$$

where

$$\cos\mu = \frac{\beta_m}{\sqrt{\beta_m^2 + 1}}, \qquad \sin\mu = \frac{1}{\sqrt{\beta_m^2 + 1}}.$$

Clearly, if we choose

$$\mu = \left(2\pi + \frac{\pi}{3} - \theta\right)/m$$

or equivalently,

$$\beta_m = \cot\frac{7\pi - 3\theta}{3m},$$

we have

$$c_m = \operatorname{Re}\{\alpha(\beta_m + i)^m\} = \frac{|\alpha|}{2}\left|\csc\frac{7\pi - 3\theta}{3m}\right|^m > 0.$$

The proof is completed. ∎

3. EVOLUTION EQUATIONS IN ONE SPACE DIMENSION

In this section, we first consider the following first order evolution equations in one space dimension:

$$u_t(x, t) = \sum_{j=1}^{m} a_j(t)\, D^j u(x, t) \qquad x \in R, \quad t \in R \tag{3.1}$$

where $D=(1/i)\,\partial_x$, and the coefficients $a_j(t)\in L^1_{\text{loc}}(R)$ are complex value functions.

THEOREM 3.1. *Let $u\in C(R; L^2(R))$ be a nonzero solution to* (3.1) *and suppose that*

$$\text{support of } u(\cdot, t_0)\subset(\beta_0, +\infty) \tag{3.2}$$

for some $t_0, \beta_0\in R$. Then, in order that

$$\text{support of } u(\cdot, t_1)\subset(\beta_0, +\infty) \tag{3.3}$$

for some $t_1\in R$, it is necessary and sufficient that

$$\int_{t_0}^{t_1} a_j(t)\,dt=0, \qquad \text{for all} \quad j\geqslant 2 \tag{3.4}$$

and

$$e^{(\int_{t_0}^{t_1} a_1(t)\,dt - i\beta_0)k}\hat{u}(k, t_0)\in H^2_+. \tag{3.5}$$

Remark. 3.1. Equation (3.5) is equivalent to

$$\sup_{b>0} e^{2(\beta_0-\eta)b}\int_{-\infty}^{+\infty} |e^{\delta a}h_0(a+ib)|^2\,da<+\infty$$

where $h_0(k)=e^{-\beta_0 k}\hat{u}(k, t_0)$ and

$$\int_{t_0}^{t_1} a_1(t)\,dt=\delta+i\eta.$$

Remark 3.2. Theorem 3.1 is still true if (3.2) and (3.3) are replaced by

$$\text{support of } u(\cdot, t_j)\subset(-\infty, \beta_0), \qquad j=1, 2 \tag{3.6}$$

and (3.5) is replaced by

$$e^{(\int_{t_0}^{t_1} a_1(t)\,dt - i\beta_0)k}\hat{u}(k, t_0)\in H^2_-. \tag{3.7}$$

Proof of Theorem 3.1. Without loss of generality we asume that $\beta_0=0$ and $t_1>t_0$. We prove the necessary part of the theorem by contradiction.

Suppose that the nonzero solution $u(x, t)$ satisfies assumption (3.2) and (3.3). In addition, we assume that condition (3.4) is not satisfied. Namely, for some $r>2$,

$$\lambda_r\equiv\int_{t_0}^{t_1} a_r(t)\,dt\neq 0$$

but

$$\lambda_j \equiv \int_{t_0}^{t_1} a_j(t)\, dt = 0, \qquad \text{for} \quad r < j \leqslant m.$$

Clearly, the Fourier transformation $\hat{u}(k, t)$ of the solution $u(x, t)$ solves

$$\frac{d\hat{u}(k, t)}{dt} = \left(\sum_{0 \leqslant j \leqslant m} \int_{t_0}^{t_1} a_j(t)\, dt\, k^j \right) \hat{u}(k, t)$$

and therefore

$$\hat{u}(k, t) = \exp\left\{ \sum_{0 \leqslant j \leqslant m} \int_{t_0}^{t} a_j(\tau)\, d\tau\, k^j \right\} \hat{u}(k, t_0), \qquad \text{for any} \quad t \in R. \tag{3.8}$$

Especially,

$$\hat{u}(k, t_1) = e^{(\lambda_r k^r + G(k))}\, \hat{u}(k, t_0)$$

where

$$\lambda_r = \textstyle\int_{t_0}^{t_1} a_r(t)\, dt \neq 0$$

and

$$G(k) = \sum_{j=0}^{r-1} \int_{t_0}^{t_1} a_j(t)\, dt\, k^j.$$

It follows from our assumptions that $\hat{u}(k, t_1) \in H_+^2$ and therefore

$$\sup_{b > 0} \int_{-\infty}^{+\infty} |\hat{u}(a + ib, t_1)|^2\, da < +\infty. \tag{3.9}$$

On the other hand, by Lemma 2.4, there exists a $\beta \in R$ such that

$$\eta = \operatorname{Re}\{\lambda_r (\beta + i)^r\} > 0.$$

For any $b > 0$, we have

$$\begin{aligned} I(b) &\equiv \int_{-\infty}^{+\infty} |\hat{u}(a + ib, t_1)|^2\, da \\ &= \int_{-\infty}^{+\infty} |e^{(\lambda_r (a+ib)^r + G(a+ib))}|^2\, |\hat{u}(a + ib, t_0)|^2\, da \\ &\geqslant \int_{\beta b}^{\beta b + 1} e^{2 \operatorname{Re}(\lambda_r (a+ib)^r) + 2 \operatorname{Re} G(a+ib)}\, |\hat{u}(a + ib, t_0)|^2\, da. \end{aligned}$$

Note that

$$f(a+ib) \equiv 2\operatorname{Re}(\lambda_r(a+ib)^r) + 2\operatorname{Re} G(a+ib)$$

is a polynomial about a and b of degree r with real constant coefficients. For each term in $f(a, b)$, if the coefficient is positive, we replace a by βb; if the coefficient is negative, we replace a by $\beta b + 1$. Then we have

$$e^{f(a,b)} \geqslant \exp\{2\operatorname{Re}(\lambda_r(\beta+i)^r)\, b^r + g(b)\} \geqslant e^{2\eta b^r + g(b)}$$

for any $a \in (\beta b, \beta b + 1)$ where $g(b)$ is a polynomial about b of degree $r-1$.
Thus,

$$\begin{aligned} I(b) &\geqslant e^{2\eta b^r + g(b)} \int_{\beta b}^{\beta b+1} |\hat{u}(a+ib, t_0)|^2\, da \\ &\geqslant e^{2\eta b^r + g(b)} \left| \int_{\beta b}^{\beta b+1} \int_0^{+\infty} e^{i(a+ib)x}\, u(x, t_0)\, dx\, da \right|^2 \\ &= e^{2\eta b^r + g(b)} \left| \int_0^{+\infty} u(x, t_0) \int_{\beta b}^{\beta b+1} e^{i(a+ib)} x\, da\, dx \right|^2 \\ &= e^{2\eta b^r + g(b)} \left| \int_0^{+\infty} e^{-bx + i\beta x} p(x)\, dx \right|^2 \end{aligned}$$

where

$$p(x) = \frac{e^{ix} - 1}{ix}\, u(x, t_0).$$

It is easy to see that

$$\int_0^{+\infty} e^{-bx + i\beta bx} p(x)\, dx = h((\beta + i) b)$$

where

$$h(z) = \int_{-\infty}^{+\infty} e^{izx} p(x)\, dx$$

is a Hardy function in the class of H_+^2. If $u(x, t_0) \neq 0$ in $L^2(R)$, then $p(x) \neq 0$ in $L^2(R)$ and by Lemma 2.3, there exist a $\delta > 0$ and a sequence of $\{b_n\}$ with $b_n \to +\infty$ such that

$$|h((\beta + i)\, b_n)| \geqslant e^{-\delta b_n}$$

for $n = 1, 2, ...$ Thus if $u(\cdot, t_0) \neq 0$ in $L^2(R)$, we have

$$I(b_n) \geqslant e^{2\eta b_n^r + g(b_n)}\, e^{-2\delta b_n}, \qquad \text{for } n = 1, 2, ...$$

which is a contradiction with (3.9) since the right side of the inequality tends to $+\infty$ as $n \to +\infty$. We must have

$$u(x, t_0) = 0 \quad \text{in} \quad L^2(R).$$

Then it follows from (3.8) that

$$u(x, t) = 0 \quad \text{in} \quad L^2(R) \quad \text{for any} \quad t \in R.$$

This is a contradiction with our assumption that u is a nonzero solution. Thus, in order that $u(x, t_1)$ is supported on the interval $(0, +\infty)$, it is necessary that

$$\int_{t_0}^{t_1} a_j(t)\, dt = 0, \qquad \text{for all} \quad j \geq 2.$$

As for (3.5), it follows from Lemma 2.1 and our assumption that $u(x, t_1)$ is supported on the interval $(\beta_0, +\infty)$.

The sufficience part of the theorem follows easily from the definition of Hardy function and Lemma 2.1. The proof is completed. ∎

COROLLARY 3.1. *Let*

$$\lambda(s, t) = \max_{2 \leq j \leq m} \left| \int_t^s a_j(\tau) dt \right|.$$

Suppose that

$$\lambda(s, t) \neq 0 \qquad \textit{for any} \quad s, t \in R \textit{ with } s \neq t. \tag{3.10}$$

Then any nonzero solution $u(x, t) \in C(R; L^2(R))$ of evolution equation (3.1) *can have support on a right horizontal half line* (*or a left horizontal half line*) *in the x-t plane for at most one moment.*

Proof. It follows directly from Theorem 3.1. ∎

COROLLARY 3.2. *Let*

$$\gamma(t) = \sum_{j=2}^{m} |a_j(t)|$$

and assume that $\gamma(t)$ does not vanish on any open interval in R. If $u \in C(R; L^2(R))$ is a solution to (3.1), *then $u(x, t)$ cannot vanish in any open subset in the x-t space unless it vanishes identically.*

Proof. Suppose that there exist $a_1 < a_2$ and $t_1 < t_2$ such that

$$u(x, t) = 0, \qquad \text{for} \quad (x, t) \in (a_1, a_2) \times (t_1, t_2).$$

For $t_1 < t < t_2$, we define

$$v(x, t) = \begin{cases} u(x, t) & \text{for} \quad x > \frac{a_1 + a_2}{2} \\ 0 & \text{for} \quad x < \frac{a_1 + a_2}{2} \end{cases}$$

It is easy to see that $v(x, t)$ solves (3.1) for $t_1 < t < t_2$. Applying Theorem 3.1 to $v(x, t)$ yields that

$$v(x, t) = 0, \qquad \text{for} \quad x \in R, \quad t \in (t_1, t_2).$$

Especially,

$$u(x, t) = 0, \qquad \text{for} \quad x > \frac{a_1 + a_2}{2}, \quad t \in (t_1, t_2).$$

Applying Theorem 3.1 again to the solution $u(x, t)$ yields

$$u(x, t) = 0, \qquad \text{for} \quad x \in R, \quad t \in R.$$

The proof is completed. ∎

We may also consider the following second order evolution equation

$$u_{tt}(x, t) = \alpha D^{(2m)} u(x, t), \qquad x \in R, \quad t \in R \tag{3.11}$$

where α is a nonzero constant and $m \geqslant 2$.

THEOREM 3.2. *Let $u \in C(R; H^m(R)) \cap C^1(R; L^2(R))$ be a solution to (3.11). If there exist $t_0 < t_1$ and a $\beta \in R$ such that*

$$\textit{support of } u(\cdot, t_j) \subset (\beta, +\infty), \qquad j = 0, 1 \tag{3.12}$$

and

$$\textit{support of } u_t(\cdot, t_0) \subset (\beta, +\infty), \tag{3.13}$$

then

$$u(x, t) \equiv 0 \qquad \textit{for any} \quad x \in R, \quad t \in R.$$

Remark 3.3. Theorem 3.2 is still true if assumptions (3.12) and (3.13) is replaced by

$$\text{support } u(\cdot, t_j) \subset (-\infty, \beta), \qquad j = 0, 1$$

and

$$\text{support } u_t(\cdot, t_0) \subset (-\infty, \beta).$$

Proof of Theorem 3.2. Without loss of generality, we assume that $\beta = 0$ and

$$u(x, t_0) = q(x), \qquad u_t(x, t_0) = p(x).$$

Let $\hat{u}(k, t)$ be the Fourier transform of the solution $u(x, t)$. Then

$$\frac{d^2 \hat{u}(k, t)}{dt^2} = \alpha k^{2m} \hat{u}(k, t),$$

$$\hat{u}(k, t_0) = \hat{q}(k), \qquad \hat{u}_t(k, t_0) = \hat{p}(k).$$

Hence

$$\hat{u}(k, t) = c_1(k)\, e^{-\mu k^m t} + c_2(k)\, e^{\mu k^m t}, \qquad \text{for any} \quad t \in R \tag{3.14}$$

where $\mu^2 = \alpha$,

$$c_1(k) = \frac{1}{2}\left(\hat{q}(k) - \frac{\hat{p}(k)}{\mu k^m}\right) \qquad c_2(k) = \frac{1}{2}\left(\hat{q}(k) + \frac{\hat{p}(k)}{\mu k^m}\right).$$

In particular,

$$k^m \hat{u}(k, t_1) = c_1(k)\, k^m e^{-\mu k^m t_1} + c_2(k)\, k^m e^{\mu k^m t_1}.$$

Note that

$$k^m \hat{u}(k, t_1) = \widehat{D^m u}(k, t_1),$$
$$k^m c_2(k) = \hat{f}(k)$$

and

$$k^m c_2(k) = \hat{g}(k)$$

where

$$f(x) = \frac{1}{2}\,(D^m q(x) - \mu p(x)), \qquad g(x) = \frac{1}{2}\,(D^m q(x) + \mu p(x)).$$

In addition, f, g and $D^m u(\cdot, t_1)$ belong to $L^2(R)$ and their supports are all contained in $(0, +\infty)$. This implies that $k^m \hat{u}(k, t_1)$, $\hat{f}$ and $\hat{g} \in H^2_+$ and we have

$$\sup_{b>0} \int_{-\infty}^{+\infty} |a + ib|^{2m}\, |\hat{u}(a + ib, t_1)|^2\, da < +\infty. \tag{3.15}$$

On the other hand,

$$I(b) \equiv \int_{-\infty}^{+\infty} |\hat{f}(a+ib)\, e^{-\mu(a+ib)^m t_1} + \hat{g}(a+ib)\, e^{\mu(a+ib)^m t_1}|^2\, da$$

$$\geqslant \int_{\beta_m b}^{\beta_m b+1} |\hat{g}(a+ib)\, e^{\mu(a+ib)^m t_1} + \hat{f}(a+ib)\, e^{-\mu(a+ib)^m t_1}|^2\, da$$

$$\geqslant \int_{\beta_m b}^{\beta_m b+1} |\hat{g}(a+ib)|^2\, e^{2\mu \operatorname{Re}((a+ib)^m t_1}\, da$$

$$-2\int_{\beta_m}^{\beta_m b+1} |\hat{f}(a+ib)|\, |\hat{g}(a+ib)|\, da$$

$$+\int_{\beta_m b}^{\beta_m b+1} |\hat{f}(a+ib)|^2\, e^{-2\mu \operatorname{Re}((a+ib)^m t_1)}\, da.$$

By Lemma 2.4, for $m \geqslant 2$, there exists a $\beta_m \in R$ such that

$$\operatorname{Re}\{\mu(\beta_m + i)^m\} = \delta_m > 0.$$

As in the proof of Theorem 3.1, we have

$$e^{2\operatorname{Re}(\mu(a+ib)^m t_1)} \geqslant e^{2\delta_m b^m t_1 + F(b) t_1}, \qquad \text{for} \quad \beta_m b \leqslant a \leqslant \beta_m + 1$$

where $F(b)$ is a real polynomial of b with degree $m-1$. Hence

$$I(b) \geqslant \exp\{2\delta_m\, b^m t_1 + F(b)\, t_1\} \int_{\beta_m b}^{\beta_m b+1} |\hat{g}(a+ib)|^2\, da$$

$$-2\int_{\beta_m b}^{\beta_m b+1} |\hat{f}(a+ib)|\, |\hat{g}(a+ib)|\, da$$

$$\geqslant \exp\{2\delta_m\, b^m t_1 + F(b)\, t_1\} \int_{\beta_m b}^{\beta_m b+1} |\hat{g}(a+ib)|^2\, da$$

$$-\int_{-\infty}^{+\infty} |\hat{f}(a+ib)|^2\, da - \int_{-\infty}^{+\infty} |\hat{g}(a+ib)|^2\, da$$

However, if $g \neq 0$ in $L^2(R)$, the same argument as in the proof of Theorem 3.1 implies that there exists a $\delta > 0$ and a sequence of $\{b_n\}$ with $b_n \to \infty$ such that

$$I(b_n) \geqslant e^{2\delta_m b_n^m t_1 + F(b_n) t_1} e^{-\delta b_n} - \int_{-\infty}^{+\infty} |\hat{f}(a+ib_n)|^2\, da$$

$$-\int_{-\infty}^{+\infty} |\hat{g}(a+ib_n)|^2\, da.$$

This is in contradiction with (3.15) since the right side of the above inequality tends to $+\infty$ as $n \to +\infty$. Therefore, we must have

$$g(x) = \frac{1}{2}(D^m q(x) + \mu p(x)) \equiv 0, \qquad \text{for any} \quad x \in R. \tag{3.16}$$

Similarly, if we choose $\beta_m \in R$ such that

$$\operatorname{Re}\{-\mu(\beta_m + i)^m\} = \delta_m > 0,$$

then we have

$$f(x) = \frac{1}{2}(D^m q(x) - \mu p(x)) \equiv 0, \qquad \text{for any} \quad x \in R. \tag{3.17}$$

Thus

$$q(x) = p(x) \equiv 0, \qquad \text{for any} \quad x \in R.$$

As a consequence,

$$u(x, t) \equiv 0, \qquad \text{for any} \quad t \in R, \quad x \in R.$$

The proof is completed. ∎

Remark 3.4. Theorem 3.2 is not true for $m = 1$. We can see this by simply checking the wave equation:

$$u_{tt}(x, t) - u_{xx}(x, t) = 0, \qquad \text{for any} \quad x, t \in R.$$

COROLLARY 3.3. *If $y \in C(R : H^m(R))$ $(m \geqslant 2)$ is a solution of* (3.11) *and $u(x, t)$ vanishes in an open subset in the x-t space, then $u(x, t)$ vanishes everywhere in the x-t space.*

Proof. It follows from the same argument as the proof of Corollary 3.1.

4. EVOLUTION EQUATIONS IN n SPACE DIMENSIONS

In this section, let $n \geqslant 1$ be an integer. Denote by R^n the product of n copies the real line R and by C^n the product of n copies of the complex plane C. We use $x = (x_1, x_2, ..., x_n)$ to denote an element in R^n and $k = (k_1, k_2, ..., k_n)$ to denote an element in C^n. By N^n we denote the subset of R^n consisting of elements $p = (p_1, p_2, ..., p_n)$ whose components p_j are nonnegative integer. For $p \in N^n$, we define

$$|p| = p_1 + p_2 + \cdots + p_n.$$

If $x \in R^n$ (or $k \in C^n$),

$$x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \qquad (k^p = k_1^{p_1} k_2^{p_2} \cdots x_n^n).$$

We say that $\vec{a} \in R^n$ is a unit direction if

$$\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} = 1.$$

For any unit direction $\vec{a}$, we define

$$\Gamma_{\vec{a}} = \left\{ x \in R^n, \sum_{j=1}^{n} x_j a_j = 0 \right\},$$

$$\Gamma_{\vec{a}}^{+} = \left\{ x \in R^n, \sum_{j=1}^{n} x_j a_j > 0 \right\}$$

and

$$\Gamma_{\vec{a}}^{-} = \left\{ x \in R^n, \sum_{j=1}^{n} x_j a_j < 0 \right\}.$$

As usual,

$$D_j = \frac{1}{i} \partial_{x_j}, \qquad 1 \leqslant j \leqslant n.$$

Consider the following time dependent differential operator

$$P(t, D) = \sum_{|p| \leqslant m} a_p(t) D^p, \qquad m \geqslant 2$$

where the coefficients $a_p(t) \in L^1_{\mathrm{loc}}(R, L^2(R^n))$ are complex value function.
We define

$$G(t_0, t_1, D) = \sum_{|p| \leqslant m} \int_{t_0}^{t_1} a_p(t)\, dt\, D^p.$$

DEFINITION. Let $t_0 \in R$ be fixed and assume that $\vec{a} \in R^n$ is a unit direction. We say that $P(t, D)$ is a differential operator of order j ($1 \leqslant j \leqslant m$) at time t_1 in the direction $\vec{a}$ if there is a rotation of coordinates

$$z = Ay$$

such that the direction of z_1-axis is same as that of $\vec{a}$ and $G^*(t_0, t_1, z) = G(t_0, t_1, Az) = G^*(t_0, t_1, z_1, z')$ ($z' = (z_2, z_3, ..., z_n)$) is a polynomial of z_1 with degree j for $z' \in R^{n-1}$ a.e.

THEOREM 4.1. *Let $t_0 \in R$ and assume that $\vec{a} \in R^n$ is a unit direction. Suppose that $u(x, t) \in C(R; L^2(R^n))$ is a nonzero solution of*

$$u_t = P(t, D)u \tag{4.1}$$

and

$$u(x, t_0) = 0, \qquad \text{for any} \quad x \in \Gamma_{\vec{a}}^+, \tag{4.2}$$

then in order that

$$u(x, t_1) = 0, \qquad \text{for any} \quad x \in \Gamma_{\vec{a}}^+, \tag{4.3}$$

it is necessary that the differential operator $P(t, D)$ is of degree $j \leqslant 1$ in the direction $\vec{a}$ at time t_1.

Remark 4.1. Theorem 4.1 is still true if (4.2) and (4.3) are replaced by

$$u(x, t_j) = 0, \qquad \text{for any} \quad x \in \Gamma_{\vec{a}}^-, \quad j = 1, 2.$$

or

$$u(x, t_j) = 0, \qquad \text{for any} \quad x \in x_0 + \Gamma_{\vec{a}}^+, \quad j = 1, 2$$

where $x_0 \in R^n$.

Proof of Theorem 4.1. Without loss of generality, we assume that $\vec{a} = (1, 0, ..., 0)$ and

$$G(t_0, t_1, k) = G(t_0, t_1, k_1, k') = \sum_{l=0}^{r} b_l(k')\, k_1^l$$

where $r \leqslant m$ and $k' = (k_2, k_3, ..., k_n) \in C^{n-1}$. Besides, we assume that (4.2) and (4.3) are satisfied by the nonzero solution $u(x, t)$.

Let $v(x_1, k')$ be the Fourier transform of $u(x, t)$ with respect to $x' = (x_2, ..., x_n)$. Then, $v(x_1, k')$ solves

$$\frac{\partial v(x_1, k', t)}{\partial t} = \sum_{|p| \leqslant m} a_p(t)(k')^{p'}\, D_{x_1}^{p_1} v(x_1 k', t)$$

for any $k' \in C^{n-1}$ where $p = (p_1, p')$ and $D_{x_1} = (1/i)(\partial/\partial x_i)$. Thus

$$\hat{n}(k_1, k', t_1) = \exp\left\{\sum_{l=0}^{r} b_l(k')\, k_1^l t_1\right\} \hat{v}(k_1, k', t_0).$$

It follows from assumptions (4.2) and (4.3) that

$$\text{the support of } v(\cdot, k', t_j) \subset (0, +\infty), \qquad \text{for any} \quad k' \in C^{n-1}, \quad j = 1, 2$$

Then, Theorem 3.1 yields that

$$b_l(k') = 0, \qquad \text{for any} \quad l \geqslant 2 \qquad \text{and} \qquad 0 \neq k' \in C^{n-1}.$$

That is to say that $P(t, D)$ is a differential operator of degree no greater than 1 in the direction $\vec{a}$ at time t_1. The proof is completed. ▮

COROLLARY 4.1. *Let $u(x, t) \in C(R; L^2(R))$ be a nonzero solution of* (4.1) *and $u(x, t)$ has compact support in the space R^n at time $t = t_0$. Then in order that $u(x, t)$ has compact support in the space R^n at another time $t = t_1$, it is necessary that*

$$\int_{t_0}^{t_1} a_p(t)\, dt = 0, \qquad \textit{for all} \quad p \in N^n \textit{ with } |p| \geqslant 2.$$

Proof. If the solution $u(x, t)$ has compact support at time $t = t_0$ and $t = t_1$, then for any unit direction $\vec{a} \in R^n$, there is a $x_0 \in R^n$ such that

$$\text{support of } u(\cdot, t_j) \subset x_0 + \Gamma_{\vec{a}}^{+}, \quad j = 1, 2.$$

It follows from Theorem 4.1 that the operator $P(t, D)$ is of degree no greater than 1 in the direction $\vec{a}$ at time $t = t_1$. Since $\vec{a}$ is arbitrary, $P(t, D)$ is of degree no greater than 1 in any direction at time $t = t_1$, which is equivalent to say that

$$\int_{t_0}^{t_1} a_p(t)\, dt = 0, \qquad \text{for any} \quad p \in N^n \text{ with } |p| \geqslant 2.$$

The proof is completed. ▮

Remark. 4.2. As a consequence of the above Corollary, all evolution equations of form (4.1) have infinite propagation speed unless the operator $P(t, D)$ is a first order differential operator. On the other hand, the only possible evolution equations of form (4.1) which has finite speed of propagation is the equation in which the operator $P(t, D)$ is a first differential operator (see [13] and [22]).

In general, Theorem 3.1 and 4.1 are not true for higher order evolution equations. To see this, consider the following evolution equation:

$$u_{tt} - u_{xt} - u_{xxt} + u_{xxx} = 0, \qquad x \in R, \quad t \in R$$

where subscripts denote partial differentiation. It is easy to check that if $f(y)$ is a smooth function, then $u(x, t) = f(x - t)$ solves the above evolution equation. Thus, if the function $f(y)$ has compact support, then the solution $u(x, t) = f(x - t)$ has compact support for any $t \in R$.

However, for the rth order evolution equation of the following form

$$\prod_{j=1}^{r} (\partial_t + P_j(t, D))\, u(x, t) = 0, \qquad x \in R^n, \quad t \in R \tag{4.4}$$

where

$$P_j(t, D) = \sum_{|p| \leqslant m_j} a_{p,j}(t)\, D^p,$$

with $a_{p,j}(t) \in L^1_{\mathrm{loc}}(R)$, it is easy to obtain the following unique continuation results from Corollary 4.1 and Corollary 3.2.

THEOREM 4.2. (i) *Suppose that $n \geqslant 1$ and each $P_j(t, D)$ is a differential operator of order no less than 2. Then any solution $u(x, t) \in C^{r-1}(R; H^{m-m_1}(R))$ of (4.4) cannot have compact support in the space R^n at two different moments unless it is identically zero.*

(ii) *Suppose $n = 1$ and the assumption of Corollary 3.2 is satisfied for each $P_j(t, D)$. If $u(x, t) \in C^{r-1}(R; H^{m-m_1}(R))$ is a solution of equation (4.4) and vanishes in an open subset of x-t space, then $u(x, t)$ vanishes everywhere in the x-t space.*

The more interesting evolution equations to study are nonlinear evolution equations and linear evolution equations with variable coefficients (not only time dependent, but also space variable dependent). However, the situation may become quite different for these equations.

As one knows that the parabolic equation $u_t - u_{xx} = 0$ possesses infinite speed of propagation, but some nonlinear parabolic equation has finite speed of propagation, i.e., if the initial state of the solution has compact support, then the solution has compact support for any time later (see [3] and [4]). Some nonlinear parabolic equations have even more interesting property that the support of nonnegative solutions instantaneously shrinks to be compact although its initial state does not have compact support (see [1] and [3]). Thus those equations do not have the unique continuation properties described above in the paper.

On the other hand, we have proved recently that the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \qquad x \in R, \quad t \in R$$

and the nonlinear Schrödinger equation

$$u_t + iu_{xx} + |u|^2 u = 0, \qquad x \in R, \quad t \in R$$

have the same unique continuation property as that of their linear counterparts. More precisely, the solutions of the Korteweg-de Vries equation and the nonlinear Schrödinger equation cannot vanish on two different horizontal half lines in the x-t space unless they are identically zero (see [20] and [21]).

APPENDIX

Proof of Lemma 2.2. Consider the linear fractional transformation

$$z=\frac{w-i}{w+i}$$

which maps the upper half w-plane into a unit disk in the z-plane and maps the curve $\{w=s(\tau);\ 0\leqslant\tau<+\infty\}$ to a continuous curve $\{z=s^*(\tau);\ 0\leqslant\tau<+\infty\}$ in the unit disk in the z-plane with

$$\lim_{\tau\to+\infty} s^*(\tau)=1.$$

Let

$$H(z)=h\left(i\frac{1+z}{1-z}\right), \qquad |z|<1.$$

Then $H(z)$ is a Hardy function in the unit disk. Consider its canonical factorization (cf. [15])

$$H(z)=B(z)\,S(z)\,Q(z)$$

where $B(z)$ is Blaschke product

$$B(z)=z^k\prod_n\frac{z_n-z}{1-\bar{z}_n z}\frac{|z_n|}{z},$$

z_n are zeros of $H(z)$ in the unit disk,

$$S(z)=\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\mu(\phi)\right\}$$

is a singular inner function (μ is a positive singular measure, possibly 0) and

$$Q(z)=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\phi}+z}{e^{i\phi}-z}\log|H(e^{i\phi})|\,d\phi\right\}$$

is an outer function. If

$$z = re^{i\psi},$$

then

$$\mathrm{Re}\left[\frac{e^{i\phi}+z}{e^{i\phi}-z}\right] = P_r(\psi-\phi),$$

the Poisson kernel, and

$$P_r(\psi-\phi) < \frac{1+r}{1-r}$$

for any ψ, ϕ and $0<r<1$. Thus

$$\log|S(x)| \geqslant -\frac{1+r}{1-r}\,\|\mu\|$$

($\|\mu\| = \mu(-\pi, \pi)$) and

$$\log|Q(z)| \geqslant -\frac{1+r}{1-r}\int_{-\pi}^{\pi} \log^{-}|H(e^{i\phi})|\,d\phi$$

where $|z| = r < 1$, $\log = \log^{+} - \log^{-}$. There exists a constant $C>0$ such that

$$(1-r)\log|S(z)\,Q(z)| \geqslant -C, \qquad \text{for} \quad |z|<1.$$

In particular,

$$(1-|s^*(\tau)|) - \log|S(s^*(\tau))\,Q(s^*(\tau))| \geqslant -C \tag{$*$}$$

for $0 \leqslant \tau \leqslant +\infty$.

As regards the Blaschke product B, we define

$$B^*(|z|) = |z|^k \prod_n \frac{|z_n| - |z|}{1-|\bar{z}_n\,|z|}.$$

Then

$$(1-|z|)\log|B^*(|z|)| \leqslant (1-|z|)\log|B(z)|$$

since

$$\left|\frac{z_n - z}{1-\bar{z}_n z}\right| \geqslant \left|\frac{|z|-|z_n|}{1-|z_n z|}\right|$$

for $n = 1, 2, \ldots$ Especially,

$$(1-|s^*(\tau)|)\log|B^*(|s^*(\tau)|)| \leqslant (1-|s^*(\tau)|)\log|B(s^*(\tau))|.$$

From the know fact (see [18]) that

$$\overline{\lim_{|z|\to 1^-}}\,(1-|z|)\log|B^*(z)| = 0,$$

we have

$$\begin{aligned} 0 &= \overline{\lim_{|z|\to 1^-}}\,(1-|z|)\log|B^*(|z|)| \\ &= \overline{\lim_{\tau\to+\infty}}\,(1-|s^*(\tau)|)\log|B^*(|s^*(\tau)|)| \\ &\leqslant \overline{\lim_{r\to+\infty}}\,(1-|s^*(\tau)|\log|B(s^*(\tau))|. \end{aligned}$$

There exists a sequence of $\{\tau_n\}$ with $\tau_n \to +\infty$ such that

$$(1-|s^*(\tau_n)|)\log|B(s^*(\tau_n))| \geqslant -C,$$

which, together with $(*)$, implies that

$$(1-|s^*(\tau_n)|)\log|H(s^*(\tau_n))| > -C$$

for $n = 1, 2, \ldots$

On the other hand,

$$h(s(\tau_n)) = H(s^*(\tau_n)).$$

Note that

$$s^*(\tau_n) = \frac{s(\tau_n)-i}{s(\tau_n)+i}$$

Thus

$$|s^*(\tau_n)| = \sqrt{\frac{u^2(\tau_n)+(v(\tau_n)-1)^2}{u^2(\tau_n)+(v(\tau_n)+1)^2}}$$

and

$$\begin{aligned} &1-|s^*(\tau_n)| \\ &\quad = \frac{4v(\tau_n)}{u^2(\tau_n)+v^2(\tau_n)+\sqrt{[u^2(\tau_n)+(v(\tau_n)+1)^2][u^2(\tau_n+(v(\tau_n)-1)^2]}} \\ &\quad \geqslant \frac{4v(\tau_n)}{u^2(\tau_n)+v^2(\tau_n)}. \end{aligned}$$

We have

$$\log |h(s(\tau_n))| \geqslant -C\frac{u^2(\tau_n)+v^2(\tau_n)}{4v(\tau_n)}$$

for $n = 1, 2, \dots$ Hence there exists a $\delta > 0$ such that

$$|h(s(\tau_n))| > e^{-\delta(u^2(\tau_n)+v^2(\tau_n)/v^2(\tau_n))}$$

for $n = 1, 2, \dots$ The proof is completed. ∎

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